United Approach to the Universe Model and the Local Gravitation Problem

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Abstract

A united approach of the large-scale structure of a closed universe and the local spherically symmetric gravitational field is given by supposing an appropriate boundary condition. The general feature of the model obtained are the following. The universe is approximately homogeneous and isotropic on the average on large scale and is expanding at present, as described by the standard model; while locally, the small exterior region of a star started long ago to contract, as expected by the gravitational collapse theory.

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1 Introduction

In the traditional approaches of the universe model and the local gravitation problem, the two subjects are alone treated respectively. In various universe models people only consider the average structure on large scale; the distribution of matter is imagined as homogenized or smoothed. A galaxy is nothing but a perfect fluid element. On the other hand, when people investigate local gravitation problem such as the spherically symmetrical gravitational field, they often regard the region as asymptotically flat spacetime, as if there is no remain matter in the universe. True, such a treatment has its reasons: (i) the model is simple, and it does not impair the knowledge of the large-scale structure of the universe to ignore its minor details. (ii) For a local gravitation problem, the Birkhoff's theorem [1] seems to be the basis for leaving the remaining matter of the universe out of consideration. However, no matter how reasonable it is apparently, the asymptotically flat behavior of the solution of local gravitational field does not be in tune with a closed universe model, while the real universe most likely is closed.

The cosmological redshift shows that the distances between galaxies are increasing, that is, the universe is expanding on the scale of the space between galaxies at present. On the other hand, the existing main form of the cosmic matter is the celestial body, some of which are very compact. This shows that in some very small areas of the expanding universe the space has begun to contract long ago. To our knowledge, there is yet no model to treat the two problems concurrently at present. In the present article we try to give, by means of the investigation a spherically symmetric gravitational field in an expanding closed universe, a united approach to the universe model and a local gravitational problem.

In order to make it practical, we shall adopt a compromise proposal, which can be called "homogeneous by areas model" of matter distribution, that is, the universe can be divided into a few spherically symmetric parts with the same symmetric center, and in each part the matter density is a constant.

2 The Gravitational Field Equations and Their General Solutions

Imagine in a closed Friedmann universe a local dense-matter area appears owing to a certain kind of mechanism. For simplicity,we assume that its density is a constant. Of

course, around the dense-matter area a spherically symmetric rare-matter or vacuum area may appear. Such a matter distribution model is our start point. In this case, the 3-dimension spacelike hypersurface, $t =$ constant, is a rotational hypersurface embedded in a 4-dimension flat spacetime. In comoving coordinates the line element can be written

$$ds^2 = dt^2 - U(\psi,t)d\psi^2 - V(\psi,t)\sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

where $0 \le \psi \le \pi$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. The relation between 4-dimension spherical coordinates R, ψ, θ and ϕ and the Decare coordinates x, y, z and w are

$$\begin{aligned} x &= R\sin\psi\sin\theta\cos\phi \\ y &= R\sin\psi\sin\theta\sin\phi \\ z &= R\sin\psi\cos\theta \\ w &= R\cos\psi \end{aligned} \tag{2}$$

One often introduces

$$r = \sin\psi \tag{3}$$

and rewrites (1) as the form

$$ds^2 = dt^2 - A(r,t)dr^2 - B(r,t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{4}$$

which is a well-known form [2].

However, it is necessary to emphasize that r is not a good coordinate owing to non-monodromy of the transformation (3). The same value of r corresponds to two values of ψ, i.e., two positions in the universe. This means that $A(r,t)$ and $B(r,t)$ should have two set physical solutions in principle, which corresponds to the two areas $0 \le \psi \le \pi/2$ and $\pi/2 \le \psi \le \pi$, respectively.

Consider the zero pressure perfect-fluid model

$$T^{\mu\nu} = \rho U^\mu U^\nu \tag{5}$$

where $U^r = U^\theta = U^\phi = 0$, $U^t = 1$. The Einstein's equations are [1]

$$\begin{aligned} \frac{1}{A}\Big(\frac{B''}{B} - \frac{B'^2}{2B^2} - \frac{A'B'}{2AB}\Big) - \frac{\ddot{A}}{2A} + \frac{\dot{A}^2}{4A^2} - \frac{\dot{A}\dot{B}}{2AB} &= -4\pi G\rho \\ -\frac{1}{B} + \frac{1}{A}\Big(\frac{B''}{2B} - \frac{A'B'}{4AB}\Big) - \frac{\ddot{B}}{2B} - \frac{\dot{A}\dot{B}}{4AB} &= -4\pi G\rho \\ \frac{\ddot{A}}{2A} + \frac{\ddot{B}}{B} - \frac{\dot{A}^2}{4A^2} - \frac{\dot{B}^2}{2B^2} &= -4\pi G\rho \\ \frac{\dot{B}'}{B} - \frac{\dot{B}B'}{2B^2} - \frac{\dot{A}B'}{2AB} &= 0 \end{aligned} \tag{6}$$

From the last equation of (6), it is easy to find that

$$A = \frac{B'^2}{4(1 - Kr^2)B} \tag{7}$$

here $K = K(r)$ is an arbitrary function of r. Using the firt three equations of (6) in (7) gives that

$$4\frac{\ddot{B}\dot{B}^2}{B^3} + \frac{4Kr^2}{B} = 0 \tag{8}$$

Set

$$B = S^2(r,t)r^2 \tag{9}$$

equation (8) becomes

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2}{S^2} + \frac{K}{S^2} = 0$$

from which we find

$$\dot{S}^2 = \frac{2KF}{S} - K \tag{10}$$

here $F = F(r)$ is an arbitrary function of r. The solution of equation (10) is cycloid-like

$$\begin{aligned} S &= F(1 - \cos\alpha) \\ C + t &= F(\alpha - \sin\alpha)/\sqrt{K} \end{aligned} \tag{11}$$

where $C = C(r)$ is an arbitrary function. Substituting equations (7), (9) and (11) into (6) gives the expression of density

$$\rho = \frac{3}{4\pi G}\frac{(FKr^3)'}{S^3r^3} \tag{12}$$

3 Homogeneous by Areas Model of the Distribution of Matter

Where we investigate the problem of spherically symmetric gravitational collapse of a celestial body in a closed universe, the "homogeneous by areas model" above is a suitable model. This model finds expression in the form of the density, that is

$$\rho(r,t) = \begin{cases} \rho_s(t) & 0 \le \psi \le \psi_1 & \text{(area } I\text{)} \\ 0 & \psi_1 \le \psi \le \psi_2 < \pi/2 & \text{(area } II\text{)} \\ \rho_u(t) & \psi_2 < \psi \le \pi & \text{(area} III\text{)} \end{cases} \tag{13}$$

where $\psi_1 = \arcsin r_1$, $\psi_2 = \arcsin r_2$. The joint conditions on the boundaries $\psi = \psi_1$ and $\psi = \psi_2$ is taken as that the metric $g_{\mu\nu}$ are continuous on these boundaries. Substituting the expression (13) into equation (12) and integrating gives

$$\begin{cases} K_I = \frac{4\pi G}{3}\rho_s F_I^2(1-\cos\alpha_I)^3 & (\text{area } I) \\ K_{II} = \lambda/(F_{II}r^3) & (\text{area } II) \\ K_{III} = \frac{4\pi G}{3}\rho_u F_{III}^2(1-\cos\alpha_{III})^3 & (\text{area} III) \end{cases} \tag{14}$$

where λ is the integral constant for the area II; for the area I and the area III, the integral constants have been taken as zero so that both K_I and K_{III} are finite at $r = 0$ (i.e., $\psi = 0$ and $\psi = \pi$). The joint conditions are

$$\begin{aligned} A_I(r_1,t) = A_{II}(r_1,t), \quad A_{II}(r_2,t) = A_{III}(r_2,t) \\ B_I(r_1,t) = B_{II}(r_1,t), \quad B_{II}(r_2,t) = B_{III}(r_2,t) \end{aligned} \tag{15}$$

The value of the parameter α in equation (11) decides that the scale factor $S(r,t)$ is expansive or contractive. As an additional condition, we demand that

$$\alpha_I(r_1,t) = \alpha_{II}(r_1,t), \quad \alpha_{II}(r_2,t) = \alpha_{III}(r_2,t) \tag{16}$$

The expressions (15) and (16) are determining solution conditions to $F(r)$, $K(r)$ and $C(r)$. From (15) we can infer that $S(r_1,t)$ is continuous on the boundaries and then from the first of (11) and (16) we can know

$$F_I(r_1) = F_{II}(r_1), \quad F_{II}(r_2) = F_{III}(r_2) \tag{17}$$

The second of (11) can be written as

$$t - t_0 = F_i\Big[(\alpha_i - \sin\alpha_i) - (\alpha_{i0} - \sin\alpha_{i0})\Big]/\sqrt{K_i} \tag{18}$$

here $i = I, II, III$ and $\alpha_{i0} = \alpha_i(r,t_0)$. Equations (16), (17) and (18) imply

$$K_I(r_1) = K_{II}(r_1), \quad K_{II}(r_2) = K_{III}(r_2) \tag{19}$$

Substituting (16), (17) and (19) into (14) gives

$$\begin{aligned} \lambda &= \frac{4\pi G}{3}\rho_s F_I^3(r_1) r_1^3\Big(1 - \cos\alpha_I(r_1,t)\Big)^3 \\ &= \frac{4\pi G}{3}\rho_u F_{III}^3(r_2) r_2^3\Big(1 - \cos\alpha_{III}(r_2,t_1)\Big)^3 \end{aligned}$$

From the first and the last of (14) it is also easy to see that

$$\begin{aligned}\rho_s(1-\cos\alpha_I)^3 &= \rho_{s0}(1-\cos\alpha_{I0})^3 \\ \rho_u(1-\cos\alpha_{II})^3 &= \rho_{u0}(1-\cos\alpha_{II0})^3\end{aligned} \tag{20}$$

In general, these conditions can yet not determine F_i and K_i (or α_{i0}) completely because there still exists a certain freedom of coordinate transformation in comoving coordinates. In this sense, to select the function form of F_i and α_{i0} is just to select coordinates system.

Now we construct a set of concrete solutions from the following considerations: (i) In the case $\psi_2 = \psi_1$ ($\rho_s = \rho_u$, see (20)), the universe should be homogeneous and isotropic. (ii) In the case $\psi \gg \psi_1$, α_{II0} and α_{III0} should be approximate constants and so are F_{II} and F_{III}. (iii) In the case $\psi \ll \psi_2$ both α_{I0} and α_{II0} should be larger than π, so that the scale factor in this area has been contracted, which reflects gravitational collapse of a celestial body. The first condition implies $\alpha_{I0} = \alpha_{II0} = \alpha_{III0} =$ constant, and $F_I = F_{II} = F_{III} =$ constant. We try to choose the following forms of α_{i0} and F_i

$$\alpha_{i0} = \beta + \frac{r_2 - r_1}{r_2 + r_1}\delta e^{-\psi/\sqrt{\psi_1\psi_2}} \tag{21}$$

$$F_i = a - \frac{r_2 - r_1}{r_2 + r_1} b e^{-\psi/\sqrt{\psi_1\psi_2}} \tag{22}$$

where β, δ, a and b are constants. Obviously, the last of (15) is satisfied because of the conditions (16), (17) and (19). In order to make the first of (15) held, we differentiate (18) with respect to r and substitute into (14). So doing, we obtain

$$\begin{aligned}(1-\cos\alpha_I)\alpha_I' &= \Big\{(1-\cos\alpha_{I0}) + \frac{3\sin\alpha_{I0}}{2(1-\cos\alpha_{I0})}\Big[(\alpha_I - \\ &\quad \sin\alpha_I) - (\alpha_{I0} - \sin\alpha_{I0})\Big]\Big\}\alpha_{I0}' \\ (1-\cos\alpha_{II})\alpha_{II}' &= (1-\cos\alpha_{II0})\alpha_{II0}' + \frac{3(F_{II}r)'}{2F_{II}r}\Big[(\alpha_{II} - \\ &\quad \sin\alpha_{II}) - (\alpha_{II0} - \sin\alpha_{II0})\Big] \\ (1-\cos\alpha_{III})\alpha_{III}' &= \Big\{(1-\cos\alpha_{III0}) + \frac{3\sin\alpha_{III0}}{2(1-\cos\alpha_{III0})}\Big[(\alpha_{III} - \\ &\quad \sin\alpha_{III}) - (\alpha_{III0} - \sin\alpha_{III0})\Big]\Big\}\alpha_{III0}'\end{aligned} \tag{23}$$

The continuity of α on the boundaries gives

$$\frac{(F_{II}r)'}{F_{II}r}\Big|_{r_1} = -\frac{\sin\alpha_{I0}}{1-\cos\alpha_{I0}}\alpha_{I0}'\Big|_{r_1}$$

$$\left.\frac{(F_{II}r)'}{F_{II}r}\right|_{r_2} = -\frac{\sin\alpha_{III0}}{1-\cos\alpha_{III0}}\alpha'_{III0}\Big|_{r_2} \tag{24}$$

or equivalent forms

$$\begin{aligned}
\frac{b\omega_1}{(a-b\omega_1)\sqrt{\psi_1\psi_2}}\frac{1}{\sqrt{1-r_1^2}}+\frac{1}{r_1} &= \frac{\delta\omega_1\sin(\beta+\delta\omega_1)}{\Big[1-\cos(\beta+\delta\omega_1)\Big]\sqrt{\psi_1\psi_2}}\frac{1}{\sqrt{1-r_1^2}} \\
\frac{b\omega_2}{(a-b\omega_2)\sqrt{\psi_1\psi_2}}\frac{1}{\sqrt{1-r_2^2}}+\frac{1}{r_2} &= \frac{\delta\omega_2\sin(\beta+\delta\omega_2)}{\Big[1-\cos(\beta+\delta\omega_2)\Big]\sqrt{\psi_1\psi_2}}\frac{1}{\sqrt{1-r_2^2}}
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\omega_1 &= \frac{r_2-r_1}{r_2+r_1}e^{-\sqrt{\psi_1}/\sqrt{\psi_2}} \\
\omega_2 &= \frac{r_2-r_1}{r_2+r_1}e^{-\sqrt{\psi_2}/\sqrt{\psi_1}}
\end{aligned}$$

If we assign any two of the four constants, say β and a (initial conditions), the other two can be determined by (25), and a set of uniquely definite solutions is thus found.

4 Disussion

From (21) and (22) it is easy to see that if $r_2 = r_1$ then $\alpha_{i0} = \beta$ and $F_i = a$, which exactly is the homogeneous and isotropic Friedmann universe model. In general, where $\psi \gg \psi_1$ and $\psi \gg \psi_2$, we have $\alpha_{III0} \approx \beta$ and $F_{III} \approx a$, which shows that the spacetime far away from the star tends to become homogeneous and isotropic. In the place where $\psi \ll \psi_2$ We have

$$\alpha_{I0} \approx \beta + \frac{r_2-r_1}{r_2+r_1}\delta \tag{26}$$

$$F_I \approx a - \frac{r_2-r_1}{r_2+r_1}b \tag{27}$$

The existing observational data show that $\beta \approx \pi/2$. If $(r_2-r_1)/(r_2+r_1)\delta > \pi/2$, then the celestial body is collapsing.

In the traditional treatment of the problem of the spherically symmetric gravitational collapse, the exterior region of the star is considered infinite and empty [2, 3, 4]. The surface of the star is only boundary. A puzzle arose out of this treatment, which we arrate as follows. Suppose the surface of a star with constant density is described by the equation

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1 \tag{28}$$

in a certain coordinate system. Under the transformations

$$x' = x, \quad y' = y, \quad z' = az/b \tag{29}$$

The surface equation becomes

$$x'^2 + y'^2 + z'^2 = a^2 \tag{30}$$

in the new coordinate system. Then which symmetry has the gravitational field, axisymmetric or spherically symmetric? In our new treatment there is no such a puzzle owing to the existence of the second boundary.

Birkhoff's theorem states that the solutions of spherically symmetric and vacuum gravitational field equations must be Schwartzschild. Hence the metric of the area II should be able to become Schwartzschild form by means of a certain transformation. Such a transformation has been found, which is

$$\begin{aligned}
\bar{r} &= F_{II} r (1 - \cos\alpha_{II}) \\
\bar{t} &= 2GM \ln \frac{\sqrt{F_{II} r - GM} \sin\alpha_{II} + \sqrt{GM}(1 + \cos\alpha_{II})}{\sqrt{F_{II} r - GM} \sin\alpha_{II} - \sqrt{GM}(1 + \cos\alpha_{II})} \\
& \sqrt{\frac{F_{II} r - GM}{GM}} (F_{II} r + 2GM)\alpha_{II} - F_{II} r \sqrt{\frac{F_{II} r - GM}{GM}} \sin\alpha_{II} \\
& - \int \sqrt{\frac{GM}{F_{II} r - GM}} \Big[\sqrt{\frac{F_{II}^3 r^3}{GM}} (\alpha_{II0} - \sin\alpha_{II0}) \Big]' dr
\end{aligned} \tag{31}$$

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